

A dispersive estimate for the Schrödinger operator in star-shaped networks

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Abstract. We prove time decay $L^1(\mathcal{R}) \rightarrow L^\infty(\mathcal{R})$, where \mathcal{R} is an infinite star-shaped network, estimates for the Schrödinger group $e^{it(-\frac{d^2}{dx^2}+V)}$ for real-valued potentials V satisfy some regularity and decay assumptions.

Mathematics Subject Classification (2010). 34B45, 47A60, 34L25, 35B20, 35B40.

Keywords. Dispersive estimate, Schrödinger operator, nonlinear Schrödinger equation, Star-shaped network.

1 Introduction

A characteristic feature of the Schrödinger equation is the loss of the localization of wave packets during evolution, the dispersion. This effect can be measured by L^∞ -time decay, which implies a spreading out of the solutions, due to the time invariance of the L^2 -norm. The well known fact that the free Schrödinger group in \mathbb{R}^n considered as an operator family from L^1 to L^∞ decays exactly as $c \cdot t^{-n/2}$ follows easily from the explicit knowledge of the kernel of this group [16, p. 60]. For Schrödinger operators in one and three space dimensions with potentials decaying sufficiently rapidly at infinity, similar estimates have been proved in [10] for the projection of the group on the subspace corresponding to the absolutely continuous spectrum (without optimality). This approach uses an expansion in generalized eigenfunctions together with estimates developed in inverse scattering theory [9].

In this paper we derive analogous L^∞ -time decay estimates for Schrödinger equations with decaying potentials on a one dimensional star shaped network. In the following we shall outline the setting and our main results. Let us introduce some notation which will be used throughout the rest of the paper.

Let $R_i, i = 1, \dots, N$, be $N(N \in \mathbb{N}, N \geq 2)$ disjoint sets identified with $(0, +\infty)$ and put $\mathcal{R} := \cup_{k=1}^N \overline{R}_k$. We denote by $f = (f_k)_{k=1, \dots, N} = (f_1, \dots, f_N)$ the function on \mathcal{R} taking their values in \mathbb{R} and f_k is the restriction of f to R_k .

Define the Hilbert space $\mathcal{H} = \prod_{k=1}^N L^2(R_k)$ with scalar product $((u_k), (v_k))_{\mathcal{H}} = \sum_{k=1}^N (u_k, v_k)_{L^2(R_k)}$

and introduce the following transmission conditions :

$$(u_k)_{k=1, \dots, N} \in \prod_{k=1}^N C(\overline{R}_k) \text{ satisfies } u_i(0) = u_k(0) \forall i, k = 1, \dots, N, \quad (1.1)$$

$$(u_k)_{k=1, \dots, N} \in \prod_{k=1}^N C^1(\overline{R}_k) \text{ satisfies } \sum_{k=1}^N \frac{du_k}{dx}(0^+) = 0. \quad (1.2)$$

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Let $H_0 : \mathcal{D}(H_0) \rightarrow H_0$ be the linear operator of H_0 defined by :

$$\mathcal{D}(H_0) = \{(u_k) \in H^2(R_k); (u_k) \text{ satisfies (1.1), (1.2)}\},$$

$$H_0(u_k) = (H_{0,k}u_k)_{k=1,\dots,N} = (-\frac{d^2 u_k}{dx^2})_{k=1,\dots,N} = -\Delta_{\mathcal{R}}.$$

The operator H_0 defined above is self-adjoint and satisfies that his spectrum $\sigma(H_0)$ is equal to $[0, +\infty)$ (see [2] for more details).

For any $s \in \mathbb{R}$, let us denote by $L_s^1(\mathcal{R})$ the space of all complex-valued measurable functions $\phi = (\phi_1, \dots, \phi_N)$ defined on \mathcal{R} such that

$$\|\phi\|_{L_s^1(\mathcal{R})} := \int_{\mathcal{R}} |\phi(x)| \langle x \rangle^s dx = \sum_{k=1}^N \int_{R_k} |\phi_k(x)| \langle x \rangle^s dx < \infty,$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. This space is a Banach space with the norm $\|\cdot\|_{L_s^1(\mathcal{R})}$.

Let $V \in L_1^1(\mathcal{R})$. Denote by H the self-adjoint realization of the operator $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathcal{R})$ and his spectrum $\sigma(H) = [0, +\infty) \cup \{\text{a finite number of negative eigenvalues}\}$ (for more details see [6] chapter 2).

We verify that the free Schrödinger group on the star-shaped network \mathcal{R} satisfies the following dispersive estimate (see Section 3)

$$\|e^{itH_0}\|_{L^1(\mathcal{R}) \rightarrow L^\infty(\mathcal{R})} \leq C|t|^{-1/2}, \quad t \neq 0.$$

Our goal is to assume as little as possible on the potential $V = V(x)$ in terms of decay or regularity. More precisely, we prove the following theorem.

1.1 Theorem. *Let $V \in L_\gamma^1(\mathcal{R})$, with $\gamma > 5/2$ and assume that (4.36) below holds. Then for all $t \neq 0$,*

$$\|e^{itH} P_{ac}(H)\|_{L^1(\mathcal{R}) \rightarrow L^\infty(\mathcal{R})} \leq C |t|^{-1/2} \quad (1.3)$$

where C is a positive constant and $P_{ac}(H)$ is the projection onto the absolutely continuous spectral subspace.

As a consequence, we have the following $L^p - L^{p'}$ estimate.

1.2 Corollary. ($L^p - L^{p'}$ estimate)

Under the assumptions of Theorem 1.1, for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$ we have for all $t \neq 0$,

$$\|e^{itH} P_{ac}(H)\|_{L^p(\mathcal{R}) \rightarrow L^{p'}(\mathcal{R})} \leq C |t|^{-\frac{1}{p} + \frac{1}{2}}, \quad (1.4)$$

where $C > 0$ is a constant.

Moreover we have the following Strichartz estimates which have been used in the context of the nonlinear Schrödinger equation to obtain well-posedness results.

1.3 Corollary. (*Strichartz estimates*) *Let the assumptions of Theorem 1.1 be satisfied. Then for $2 \leq p, q \leq +\infty$ and $\frac{1}{p} + \frac{2}{q} = \frac{1}{2}$ we have for all t ,*

$$\|e^{itH} P_{ac}(H)f\|_{L^q(\mathbb{R}, L^p(\mathcal{R}))} \leq C \|f\|_2, \quad \forall f \in L^p(\mathcal{R}) \cap L^2(\mathcal{R}), \quad (1.5)$$

where $C > 0$ is a constant.

As a direct consequence, see [8], we have the following well-posedness result for a nonlinear Schrödinger equation with potential.

Let $p \in (0, 4)$ and suppose that V satisfies the assumptions of Theorem 1.1. Then, for any $u_0 \in L^2(\mathcal{R})$, there exists a unique solution

$$u \in C(\mathbb{R}; L^2(\mathcal{R})) \cap \bigcap_{(p,q) \text{ admissible}} L_{loc}^q(\mathbb{R}; L^p(\mathcal{R}))$$

of the equation

$$\begin{cases} iu_t - \Delta_{\mathcal{R}} u + V u \pm |u|^p u = 0, & t \neq 0, \\ u(0) = u^0, & t = 0, \end{cases} \quad (1.6)$$

and where (p, q) is called an admissible pair if (p, q) satisfies that $2 \leq p, q \leq +\infty$ and $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$.

1.4 Remark. *Another direct consequence of the dispersive estimate (1.3) or of the $L^p - L^{p'}$ (1.4) estimate is that we can construct, as in [18], the scattering operator for the nonlinear Schrödinger equation with potential.*

The paper is organized as follows. The second section deals with a counterexample which shows that the decay of the Schrödinger operator from $L^1(\mathcal{R})$ to $L^\infty(\mathcal{R})$ as $|t|$ goes to infinity is not guaranteed for all infinite networks. In section 3, we prove the dispersive estimate for the free Schrödinger operator on star-shaped networks and we give some direct applications. The expansion in generalized eigenfunctions needed for the proof of Theorem 1.1, is given in section 4. In the last section we give the proof of the main result of the paper (Theorem 1.1).

The main lines of our arguments are the following. The counter example (section 2) uses explicit formulas for eigenfunctions of the laplacian on infinite trees from [14]. The L^∞ -time decay of the free Schrödinger group on a star shaped network is reduced to the corresponding estimate on \mathbb{R} using an appropriate change of variables (section 3). The task of finding a complete family of generalized eigenfunctions for the Schrödinger operator with potential on the star shaped network is reduced to the case of the real line by separating the branches and extending the equations on \mathbb{R} with vanishing potential. The generalized eigenfunctions on \mathbb{R} resulting from techniques from [9] are then combined to families on the network by introducing correction terms to establish the transmission conditions. Using results of [9] for the real line case, we derive estimates showing the dependence of the generalized eigenfunctions on the potential. This enables us to prove a limiting absorption principle and then to derive an expansion of the Schrödinger group on the star in these generalized eigenfunctions (section 4) following [3, 4]. The proof of the L^∞ -time decay is divided in the low frequency and high frequency part, essentially following the lines of [10]. For the high frequency components, the potential appears as a small perturbation: the resolvent of the Schrödinger operator can be expanded in a Neumann type series in terms of the resolvent of the free Schrödinger operator. By inserting this in Stones formula and exchanging the integration over the frequencies and the summation of the Neumann series, one reduces the estimate to the free case. For the low frequency components one uses the expansion in generalized eigenfunctions derived in section 4, especially the qualitative knowledge of the dependence of the generalized eigenfunctions on the potential. This enables us to construct a representation of the solution as the free Schrödinger group acting on a well chosen (artificial) initial condition, which encodes the influence of the potential. Then one concludes using the results of section 3.

Our approach does not furnish optimal results, as for example the estimate in [16, p. 60] for the free Schrödinger group or the results of [5]. This is due to the fact, that the use of Neumann type series and qualitative estimates from inverse scattering theory are too rough for this purpose. We conjecture that optimal estimates could be achieved in terms of an asymptotic expansion of first order following the lines of [5], where this problem has been solved for initial conditions in energy bands for the Klein Gordon equation with constant but different potentials on a star shaped network. To deal with non constant but localized potentials (starting in the real line case), one could first derive this asymptotic expansion of first order for potentials which are characteristic functions of intervals, the corresponding spectral theory being well known (using stationary phase arguments as in [5]). Then one could approximate general potentials by step functions, inspired by [7]. This type of results should exhibit L^∞ -time decay as consequence of the asymptotic dynamics of wave packets as in [16, p. 60] and [5] and should show that these dynamics are essentially the same as for the free Schrödinger group, if the potentials are sufficiently localized.

Another perspective would be the application of the present results to the existence of global small solutions of Schrödinger equations with a nonlinearity which is sufficiently flat at zero, following [15].

Note that the general perturbation theory for semigroups [12, ch. 9, thm. 2.12, p. 502] is applicable but not useful for our purposes: it yields that the difference between the (semi-)groups generated by the Schrödinger operator with potential and the free one grows at most proportionally to t , which engulfs the time decay at infinity. Nevertheless it furnishes additional information for small t .

The Trotter product formula [16, thm. X.51, p. 245] is also applicable, but cannot establish L-infinity time decay either: it consists of an approximation of the perturbed group by long alternating compositions of values of the free Schrödinger group e^{itH_0} and the group of multiplication operators with e^{itV} but for small values of t . Thus even the explicit knowledge of the kernel of the free Schrödinger group is not useful for time-decay, because the factor $t^{-1/2}$ becomes effective only for large t .

The direct application of the variation of constants formula leads to the same phenomenon as the perturbation for semigroups: without a refined study of the superposition of the waves generated by the potential, the rough estimation of the integral term leads to a bound growing as a constant times t .

2 A counterexample

Consider the infinite network $\mathcal{R} = \cup_{n \in \mathbb{N}} e_n$, where each edge $e_n = (n, n+1)$ with the set of vertices $\mathcal{V} = \cup_{n \in \mathbb{N}} v_n$, where $v_n = \{n\}$. For a fixed sequence of positive real numbers $\alpha = (\alpha_n)_{n \in \mathbb{N}}$, we define the Hilbert space $L^2(\mathcal{R}, \alpha)$ as follows

$$L^2(\mathcal{R}, \alpha) = \{u = (u_n)_{n \in \mathbb{N}} : u_n \in L^2(e_n) \forall n \in \mathbb{N} \text{ such that } \sum_{n \in \mathbb{N}} \alpha_n \int_{e_n} |u_n(x)|^2 dx < \infty\},$$

equipped with the inner product

$$(u, v) = \sum_{n \in \mathbb{N}} \alpha_n \int_{e_n} u_n(x) v_n(x) dx, \quad \forall u, v \in L^2(\mathcal{R}, \alpha).$$

Similarly for all $k \in \mathbb{N}^*$, we set

$$H^k(\mathcal{R}, \alpha) = \{u = (u_n)_{n \in \mathbb{N}} \in L^2(\mathcal{R}, \alpha) : (u_n^{(\ell)})_{n \in \mathbb{N}} \in L^2(\mathcal{R}, \alpha) \forall \ell \in \{1, 2, \dots, k\}\},$$

where $u_n^{(\ell)}$ means the ℓ derivative of u_n with respect to x .

Now we consider the Laplace operator $-\Delta_\alpha$ (depending on α) as follows:

$$\mathcal{D}(-\Delta_\alpha) = \{u = (u_n)_{n \in \mathbb{N}} \in H^2(\mathcal{R}, \alpha) : \text{satisfying (2.7), (2.8), (2.9) below}\},$$

$$u_0(0) = 0, \tag{2.7}$$

$$u_n(n+1) = u_{n+1}(n+1), \forall n \in \mathbb{N}, \tag{2.8}$$

$$\alpha_n \frac{du_n}{dx}(n+1) = \alpha_{n+1} \frac{du_{n+1}}{dx}(n+1), \forall n \in \mathbb{N}. \tag{2.9}$$

For all $u \in \mathcal{D}(-\Delta_\alpha)$, we set

$$-\Delta_\alpha u = \left(-\frac{d^2 u_n}{dx^2}\right)_{n \in \mathbb{N}}.$$

By section 1.5 of [14], this operator is a non negative self-adjoint operator in $L^2(\mathcal{R}, \alpha)$.

Moreover in Theorem 1.13 of [14] it was shown the

2.1 Theorem. *For all $k \in \mathbb{N}^*$, $-k^2 \pi^2$ is a simple eigenvalue of $-\Delta_\alpha$ if and only if*

$$s = \sum_{n \in \mathbb{N}} \frac{1}{\alpha_n} < \infty. \tag{2.10}$$

In that case the associated orthonormal eigenvector $\varphi^{[k]} = (\varphi_n^{[k]})_{n \in \mathbb{N}}$ is given by

$$\varphi_n^{[k]} = \sqrt{\frac{2}{s}} \frac{(-1)^{(n-1)k}}{\alpha_n} \sin(k\pi(x-n)), \forall x \in e_n, n \in \mathbb{N}.$$

Now assuming that (2.10) holds, then for any $k \in \mathbb{N}^*$ we consider the solution u of the Schrödinger equation

$$\begin{cases} \partial_t u - i\Delta_\alpha u = 0, \\ u(t=0) = \varphi^{[k]}, \end{cases}$$

or equivalently solution of

$$\begin{cases} \partial_t u_n - i\partial_x^2 u_n = 0, & \text{in } e_n \times \mathbb{R}, \\ u_0(0, t) = 0, & \text{on } \mathbb{R}, \\ u_n(n+1, t) = u_{n+1}(n+1, t) & \text{on } \mathbb{R}, \forall n \in \mathbb{N}, \\ \alpha_n u'_n(n+1, t) = \alpha_{n+1} u'_{n+1}(n+1, t) & \text{on } \mathbb{R}, \forall n \in \mathbb{N}, \\ u(t=0, \cdot) = \varphi^{[k]} & \text{on } \mathcal{R}. \end{cases}$$

This solution is given by $u(t) = e^{-itk^2\pi^2} \varphi^{[k]}$. Moreover simple calculations show that

$$\|u(t)\|_{\infty, \mathcal{R}} = \sqrt{\frac{2}{s}} \sup_{n \in \mathbb{N}} \frac{1}{\alpha_n} \|\sin(k\pi(\cdot - n))\|_{\infty, e_n} = \sqrt{\frac{2}{s}} \sup_{n \in \mathbb{N}} \frac{1}{\alpha_n},$$

which is independent of t and then does not tend to zero as $|t|$ goes to infinity. On the other hand $u(t=0, \cdot)$ belongs to $L^1(\mathcal{R})$, since we have

$$\|u(t)\|_{L^1(\mathcal{R})} = \sqrt{\frac{2}{s}} \sum_{n \in \mathbb{N}} \frac{1}{\alpha_n} \|\sin(k\pi(\cdot - n))\|_{L^1(e_n)} \leq \sqrt{2s}.$$

In other words, we have proved the

2.2 Theorem. *If (2.10) holds, then the norm of the Schrödinger operator $e^{it\Delta_\alpha}$ from $L^1(\mathcal{R})$ to $L^\infty(\mathcal{R})$ does not tend to zero as $|t|$ goes to infinity.*

This counterexample shows that the decay of the norm of the Schrödinger operator from $L^1(\mathcal{R})$ to $L^\infty(\mathcal{R})$ as $|t|$ goes to infinity is not guaranteed for all infinite networks. Hence the remainder the paper is to give some examples where such a case occurs.

3 Dispersive estimate for free Schrödinger operator on star-shaped network

3.1 Theorem. *(Dispersive estimate)*

For all $t \neq 0$,

$$\|e^{itH_0}\|_{L^1(\mathcal{R}) \rightarrow L^\infty(\mathcal{R})} \leq C |t|^{-1/2}, \quad (3.11)$$

where $C > 0$ is a constant.

Proof. Let v_j , $j = 1, \dots, N$, a solution of the following problem

$$\begin{cases} \partial_t v_j = -i\partial_x^2 v_j, \mathbb{R}^+ \times \mathbb{R}^+, \\ v_j(t, 0) = v_1(t, 0), \sum_{j=1}^N \partial_x v_j(t, 0) = 0, \mathbb{R}^+, \\ v_j(0, x) = v_j^0(x), \mathbb{R}^+. \end{cases}$$

If we denote by $w_1 = \sum_{j=1}^N v_j$ and $w_j = v_j - v_1$, $\forall j = 2, \dots, N$.

Then w_1 satisfies

$$\begin{cases} \partial_t w_1 = -i\partial_x^2 w_1, \mathbb{R}^+ \times \mathbb{R}^+, \\ \partial_x w_1(t, 0) = 0, \mathbb{R}^+, \\ w_1(0, x) = \sum_{j=1}^N v_j^0(x), \mathbb{R}^+, \end{cases}$$

and w_j , $j = 2, \dots, N$, satisfies the following problem

$$\begin{cases} \partial_t w_j = -i \partial_x^2 w_j, \mathbb{R}^+ \times \mathbb{R}^+, \\ w_j(t, 0) = 0, \mathbb{R}^+, \\ w_j(0, x) = v_j^0(x) - v_1^0(x), \mathbb{R}^+. \end{cases}$$

By an odd reflexion transformation applied to w_1 , we obtain $\tilde{w}_1(t, x) = \begin{cases} w_1(t, x), x > 0, \\ -w_1(t, -x), x < 0, \end{cases}$

which verifies

$$\begin{cases} \partial_t \tilde{w}_1 = -i \partial_x^2 \tilde{w}_1, \mathbb{R}^2, \\ \tilde{w}_1(0, x) = \sum_{j=1}^N \tilde{v}_j^0(x), \mathbb{R}, \end{cases}$$

where $\tilde{v}_j^0 = \begin{cases} v_j^0(x), x > 0, \\ -v_j^0(-x), x < 0, \end{cases}$, $j = 1, \dots, N$. So, according to the dispersive estimate for Schrödinger operator on the line (see [12] or [16] for more details), we have

$$\|w_1\|_{L^\infty(\mathbb{R}^+)} \leq \|\tilde{w}_1\|_{L^\infty(\mathbb{R})} \leq C |t|^{-\frac{1}{2}} \left\| \sum_{j=1}^N \tilde{v}_j^0 \right\|_{L^1(\mathbb{R})}, \quad \forall (v_j^0) \in L^2(\mathcal{R}) \cap L^1(\mathcal{R}), \quad (3.12)$$

where $C > 0$ is a constant.

Which implies

$$\|w_1\|_{L^\infty(\mathbb{R})} \leq C |t|^{-\frac{1}{2}} \left\| \sum_{j=1}^N v_j^0 \right\|_{L^1(\mathbb{R})}, \quad \forall (v_j^0) \in L^2(\mathcal{R}) \cap L^1(\mathcal{R}).$$

With the same manner and by an even reflexion transformation applied to w_j , $j = 2, \dots, N$ i.e.

$$\tilde{w}_j(t, x) = \begin{cases} w_j(t, x), x > 0, \\ w_j(t, -x), x < 0, \end{cases}, \quad j = 2, \dots, N, \text{ we have :}$$

$$\|w_k\|_{L^\infty(\mathbb{R}_x)} \leq \|\tilde{w}_1\|_{L^\infty(\mathbb{R}_x)} \leq 2C |t|^{-\frac{1}{2}} \|v_j^0 - v_1^0\|_{L^1(\mathcal{R})}, \quad \forall (v_j^0) \in L^2(\mathcal{R}) \cap L^1(\mathcal{R}), \quad (3.13)$$

where $C > 0$ is a constant.

$$\text{Since, } v_j = w_j + v_1, \forall j = 2, \dots, N \text{ and } v_1 + \sum_{j=2}^N (w_j + v_1) = w_1 \Rightarrow v_1 = \frac{1}{N} w_1 - \frac{1}{N} \sum_{j=2}^N w_j.$$

Thus (3.12)-(3.13) imply that

$$\|v_1\|_{L^\infty(\mathbb{R}^+)} \leq \frac{4C}{N} |t|^{-\frac{1}{2}} \sum_{j=2}^n \left(\|v_j^0\|_{L^1(\mathbb{R}^+)} + \|v_1^0\|_{L^1(\mathbb{R}^+)} \right), \quad \forall (v_j^0) \in L^2(\mathcal{R}) \cap L^1(\mathcal{R}), \quad (3.14)$$

where $C > 0$ is a constant.

According to the above we have

$$\|v_1\|_{L^\infty(\mathbb{R}^+)} \leq 4C |t|^{-\frac{1}{2}} \sum_{j=1}^N \left(\|v_j^0\|_{L^1(\mathbb{R}^+)} + \|v_1^0\|_{L^1(\mathbb{R}^+)} \right), \quad (3.15)$$

and

$$\begin{aligned} \|v_j\|_{L^\infty(\mathbb{R}^+)} &\leq \|w_k\|_{L^\infty(\mathbb{R}^+)} + \|v_1\|_{L^\infty(\mathbb{R}^+)} \leq 2C |t|^{-\frac{1}{2}} \left(\|v_j^0\|_{L^1(\mathbb{R}^+)} + \|v_1^0\|_{L^1(\mathbb{R}^+)} \right) + \\ &4C |t|^{-\frac{1}{2}} \left(\sum_{j=1}^n \|v_j^0\|_{L^1(\mathbb{R}^+)} + \|v_1^0\|_{L^1(\mathbb{R}^+)} \right), \quad \forall (v_j^0) \in L^2(\mathcal{R}) \cap L^1(\mathcal{R}), \end{aligned} \quad (3.16)$$

\Rightarrow

$$\|v_j\|_{L^\infty(\mathbb{R}^+)} \leq 8C |t|^{-\frac{1}{2}} \sum_{j=1}^N \|v_j^0\|_{L^1(\mathbb{R}^+)}, \quad \forall (v_j^0) \in L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \forall j \geq 2. \quad (3.17)$$

Finally we obtain for all $t \neq 0$, $(v_j^0) \in L^2(\mathcal{R}) \cap L^1(\mathcal{R})$,

$$\|(v_j)\|_{L^\infty(\mathcal{R})} \leq 4(2N-1)C |t|^{-\frac{1}{2}} \sum_{j=1}^N \|v_j^0\|_{L^1(\mathbb{R}^+)} = 4(2N-1)C |t|^{-\frac{1}{2}} \|(v_j^0)\|_{L^1(\mathcal{R})}, \quad (3.18)$$

which implies (3.11). \square

We have the following result as a direct consequence for a dispersive estimate for free Schrödinger operator on a star-shaped network.

3.2 Corollary. ($L^p - L^{p'}$ estimate)

For $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$ we have for all $t \neq 0$,

$$\|e^{itH_0}\|_{L^p(\mathcal{R}) \rightarrow L^{p'}(\mathcal{R})} \leq C |t|^{-\frac{1}{p} + \frac{1}{2}}, \quad (3.19)$$

where $C > 0$ is a constant.

Proof. According to (3.11) we have

$$\sup_{t \neq 0} |t|^{\frac{1}{2}} \|e^{itH_0} f\|_\infty \leq C \|f\|_1, \quad \forall f \in L^1(\mathcal{R}) \cap L^2(\mathcal{R}).$$

Interpolating with the L^2 bound $\|e^{itH_0} f\|_2 = \|f\|_2$, leads to

$$\sup_{t \neq 0} |t|^{-\frac{1}{2} + \frac{1}{p}} \|e^{itH_0} f\|_{p'} \leq C \|f\|_p, \quad \forall f \in L^1(\mathcal{R}) \cap L^2(\mathcal{R}), \quad (3.20)$$

where $1 \leq p \leq 2$. It is well-known that via T^*T argument (3.20) gives rise to the class of Strichartz estimates

$$\|e^{itH_0} f\|_{L_t^q(L_x^p)} \leq C \|f\|_2, \quad \forall \frac{2}{q} + \frac{1}{p} = \frac{1}{2}, \quad 2 < q \leq +\infty, \quad 2 \leq p \leq \infty. \quad (3.21)$$

The endpoint $q = 2$ is not captured by this approach but by the approach developed by Kell and Tao in [13]. So the estimate (3.21) is valid for all $2 \leq p, q \leq +\infty$ satisfying $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$ and we have also,

$$\left\| \int_{\mathbb{R}} e^{-itH_0} F(s, \cdot) ds \right\|_{L^2(\mathcal{R})} \leq C \|F\|_{L^{q'}(\mathbb{R}, L^{p'}(\mathcal{R}))},$$

$$\left\| \int_0^t e^{i(t-s)H_0} F(s) ds \right\|_{L^q(\mathbb{R}, L^{r'}(\mathcal{R}))} \leq C \|F\|_{L^{r'}(\mathbb{R}, L^{s'}(\mathcal{R}))},$$

for all admissible pairs (q, p) and (r, s) satisfying $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$, $2 \leq q, p \leq +\infty$.

By the same way we can prove Corollary 1.3.

According to (3.21) and [8], we have for $p \in (0, 4)$, that for any $u_0 \in L^2(\mathcal{R})$ the equation

$$iu_t - \Delta_{\mathcal{R}} u \pm |u|^p u = 0, \quad t \neq 0, \quad u = u_0, \quad t = 0,$$

admits a unique solution $u \in C(\mathbb{R}, L^2(\mathcal{R}) \cap \bigcap_{(q,r) \text{ admissible}} L_{loc}^q(\mathbb{R}, L^r(\mathcal{R})))$.

The $L^2(\mathcal{R})$ -norm is conserved along the time, i.e., $\|u(t)\|_{L^2(\mathcal{R})} = \|u_0\|_{L^2(\mathcal{R})}$. \square

4 Expansion in generalized eigenfunctions

The goal of this section is to find an explicit expression for the kernel of the resolvent of the operator H on the star-shaped network defined in section 1. First we separate the branches by extending the potential of the Schrödinger operator by zero on $(-\infty, 0)$. Using [9], we construct N families of generalized eigenfunctions of the resulting N Schrödinger operators on \mathbb{R} , which we recombine on the network. This approach can be compared with the ones developed for Klein-Gordon equations in \mathcal{R} by [3, 4].

For each $j = 1, \dots, N$, we recall that R_j is identified to $(0, +\infty)$ and denote by V_j the restriction of V to R_j . Consider R_j as a subset of \mathbb{R} and denote by \tilde{V}_j the extension of V_j by 0 outside R_j .

Now according to [9] (see also [18]) for all $z \in \mathbb{C}^+ := \{z_1 \in \mathbb{C} : \Im z_1 \geq 0\}$, there exist two functions $f_{j,\pm}(z, \cdot)$ that satisfy the differential equation

$$-f_{j,\pm}''(z, x) + \tilde{V}_j(x)f_{j,\pm}(z, x) = z^2 f_{j,\pm}(z, x) \text{ on } \mathbb{R}, \quad (4.22)$$

and that have the asymptotic behaviour

$$|f_{j,\pm}(z, x) - e^{\pm izx}| \rightarrow 0 \text{ as } x \rightarrow \pm\infty. \quad (4.23)$$

According to section 1 of [9] (see also [18, p. 45]) we write

$$f_{j,\pm}(z, x) = e^{\pm izx} m_{j,\pm}(z, x),$$

to remove the oscillations of $f_{j,\pm}$ at infinity. The functions $m_{j,\pm}$ are the unique solutions of the Volterra integral equations:

$$m_{j,+}(z, x) = 1 + \int_x^{+\infty} \frac{e^{2iz(y-x)} - 1}{2iz} \tilde{V}_j(y) m_{j,+}(z, y) dy, \quad (4.24)$$

$$m_{j,-}(z, x) = 1 + \int_{-\infty}^x \frac{e^{2iz(y-x)} - 1}{2iz} \tilde{V}_j(y) m_{j,-}(z, y) dy, \quad (4.25)$$

and are called Jost functions (see [9, 17]). Recall that Lemma 1 of [9] (see also (2.5) of [18]) implies that

$$|m_{j,+}(z, x)| \leq C, \quad \forall x \in [0, \infty), z \in \mathbb{C}^+, \quad (4.26)$$

$$|m_{j,-}(z, x)| \leq 1 + C \frac{1+x}{1+|z|}, \quad \forall x \in [0, \infty), z \in \mathbb{C}^+, \quad (4.27)$$

for some $C > 0$. Accordingly as $f_{j,\pm}(z, x) = e^{\pm izx} m_{j,\pm}(z, x)$, we get

$$|f_{j,+}(z, x)| \leq C, \quad \forall x \in [0, \infty), z \in \mathbb{C}^+, \quad (4.28)$$

$$|f_{j,-}(z, x)| \leq C(1+x)e^{\Im z x}, \quad \forall x \in [0, \infty), z \in \mathbb{C}^+. \quad (4.29)$$

Property (4.23) implies the existence of functions $T_j, R_{j,\pm}, j = 1, \dots, N$, called transmission and reflection coefficients, such that

$$\begin{aligned} f_{j,+}(x, r) &\sim \frac{1}{T_j(r)} e^{irx} + \frac{R_{j,-}(r)}{T_{j,-}(r)} e^{-irx}, \quad x \rightarrow -\infty \\ f_{j,-}(x, r) &\sim \frac{1}{T_j(r)} e^{-irx} + \frac{R_{j,+}(r)}{T_{j,+}(r)} e^{irx}, \quad x \rightarrow \infty \end{aligned}$$

for $r \in \mathbb{R}$. For future purposes, for all real numbers r , we need the scattering matrix $S_j(r) \in \mathbb{C}^{2 \times 2}$ associated with (4.22) given by

$$S_j(r) = \begin{pmatrix} T_j(r) & R_{j,2}(r) \\ R_{j,1}(r) & T_j(r) \end{pmatrix}$$

and that is continuous on \mathbb{R} . According to [9], T_j has a meromorphic extension to \mathbb{C}^+ (with a finite numbers of simple poles that are non zero purely imaginary numbers) that is given by (see [9, p. 145])

$$\frac{1}{T_j(z)} = 1 - \frac{1}{2iz} \int_{-\infty}^{+\infty} \tilde{V}_j(y) m_{j,+}(z, y) dy \quad \forall z \in \mathbb{C}^+. \quad (4.30)$$

Since \tilde{V}_j has its support in $(0, +\infty)$, by remark 10 of [9] $R_{j,2}$ admits also a meromorphic extension on $\mathbb{C}^+ \setminus \mathbb{R}$ (with the same poles as the ones of T_j) that is given by (compare [9, p. 145] when z is real)

$$\frac{R_{j,2}(z)}{T_j(z)} = \frac{1}{2iz} \int_{-\infty}^{+\infty} e^{2izy} \tilde{V}_j(y) m_{j,+}(z, y) dy \quad \forall z \in \mathbb{C}^+. \quad (4.31)$$

Due to the fact that \tilde{V}_j is zero on $(-\infty, 0)$, the generalized eigenfunctions $f_{j,\pm}$ of the Schrödinger operators on the line have the following properties.

4.1 Lemma. *For all $z \in \mathbb{C}^+$, $z \neq 0$, we have*

$$f_{j,-}(z, x) = e^{-izx} \quad \forall x \leq 0, \quad (4.32)$$

$$f_{j,+}(z, x) = \frac{1}{T_j(z)} e^{izx} + \frac{R_{j,2}(z)}{T_j(z)} e^{-izx} \quad \forall x \leq 0. \quad (4.33)$$

In particular, it holds

$$f_{j,-}(z, 0) = 1, \quad (4.34)$$

$$f_{j,+}(z, 0) = \frac{1 + R_{j,2}(z)}{T_j(z)}. \quad (4.35)$$

Proof. From the expression (4.25), we directly get (4.32) and (4.34). The situation more complicated for $f_{j,+}$. Indeed from the expression (4.24), we see that

$$m_{j,+}(z, x) = 1 + \int_0^{+\infty} \frac{e^{2iz(y-x)} - 1}{2iz} \tilde{V}_j(y) m_{j,+}(z, y) dy, \quad \forall x \leq 0.$$

This is equivalent to

$$\begin{aligned} m_{j,+}(z, x) &= 1 - \frac{1}{2iz} \int_0^{+\infty} \tilde{V}_j(y) m_{j,+}(z, y) dy + \frac{e^{-2izx}}{2iz} \int_0^{+\infty} e^{2izy} \tilde{V}_j(y) m_{j,+}(z, y) dy \\ &= 1 - \frac{1}{2iz} \int_{-\infty}^{+\infty} \tilde{V}_j(y) m_{j,+}(z, y) dy + \frac{e^{-2izx}}{2iz} \int_{-\infty}^{+\infty} e^{2izy} \tilde{V}_j(y) m_{j,+}(z, y) dy, \quad \forall x \leq 0. \end{aligned}$$

Hence according to the expression of $\frac{1}{T_j(z)}$ and $\frac{R_{j,2}(z)}{T_j(z)}$ given in (4.30) and (4.31), we obtain (4.33). According to this identity we trivially have

$$f_{j,+}(z, 0) = \frac{1 + R_{j,2}(z)}{T_j(z)}.$$

□

For our next considerations, we need that

$$f_{j,+}(z, 0) \neq 0,$$

at least for all $z \in \mathbb{C}^+$ close to the real axis.

Therefore we make the following assumption:

$$1 + \int_0^{+\infty} x V_j(x) m_{x,+}(0, x) dx \neq 0, \quad \forall j = 1, \dots, N, \quad (4.36)$$

that allows to obtain the next result.

4.2 Lemma. *If the assumption (4.36) holds, then there exists $\kappa > 0$ small enough and two positive constants C_1, C_2 such that*

$$C_1 \leq |f_{j,+}(z, 0)| \leq C_2 \quad \forall z \in B_\kappa, \quad (4.37)$$

where $B_\kappa = \{z_1 \in \mathbb{C}^+ : 0 \leq \Im z_1 \leq \kappa\}$.

Proof. Recall that

$$f_{j,+}(z, 0) = \frac{1 + R_{j,2}(z)}{T_j(z)}.$$

By (4.30) and (4.31) we see that (see property IV of Theorem 1 in [9], p. 149) there exist $R, C > 0$ such that

$$|T_j(z) - 1| + |R_{j,2}(z)| \leq \frac{C}{|z|}, \forall |z| > R. \quad (4.38)$$

Hence (4.37) holds for all $|z| > R_0$, with R_0 large enough.

Now for $|z|$ small, we remark that $\frac{1+R_{j,2}(z)}{T_j(z)}$ is different from zero for all $z \in \mathbb{R} \setminus \{0\}$ by using the properties II and V of Theorem 1 in [9]. Furthermore using (4.30) and (4.31), one easily checks that

$$\lim_{z \rightarrow 0} \frac{1 + R_{j,2}(z)}{T_j(z)} = 1 + \int_0^{+\infty} t V_j(t) m_{j,+}(0, t) dt. \quad (4.39)$$

Consequently our assumption guarantees that the continuous function $f_{j,+}(\cdot, 0)$ is different from zero on the whole compact $[-R_0, R_0]$ and therefore (4.37) holds for all real numbers $z \in [-R_0, R_0]$. By the continuity of $f_{j,+}(\cdot, 0)$ on $B_{\delta'}$ for δ' small enough, we deduce that (4.37) holds for all $z \in B_\kappa \cap \{z_1 \in \mathbb{C} : \Re z_1 \in [-R_0, R_0]\}$, by choosing κ small enough. \square

The assumption (4.36) is quite realistic and is satisfied by an extremely large choice of potentials. Let us list some specific examples.

4.3 Lemma. 1. *In the generic case, namely if*

$$\int_0^{+\infty} V_j(x) m_{j,+}(0, x) dx \neq 0,$$

then we have

$$1 + \int_0^{+\infty} x V_j(x) m_{j,+}(0, x) dx \neq 0, \quad (4.40)$$

if V_j is non negative or if

$$\int_0^{+\infty} x |V_j(x)| dx \leq \ln 2.$$

2. *In the exceptional case, namely if*

$$\int_0^{+\infty} V_j(x) m_{j,+}(0, x) dx = 0,$$

then (4.40) always holds.

Proof. In the exceptional case, by Theorem 1 of [9], there exist two constants $c, C \in (0, 1)$ such that

$$c \leq |T_j(r)| \text{ and } |R_{j,2}(r)| \leq C, \forall r \in \mathbb{R}.$$

Hence

$$\lim_{\substack{r \rightarrow 0 \\ r \in \mathbb{R}}} \left| \frac{1 + R_{j,2}(r)}{T_j(r)} \right| \geq \frac{1 - C}{c},$$

which implies that (4.40) holds.

In the generic case and if V_j is non negative, then $m_{j,+}(0, \cdot)$ is a non negative function and therefore (4.40) directly holds.

In the generic case and if V_j has no sign, then the considerations of Lemma 1 of [9] shows that

$$|m_{j,+}(0, 0)| \geq 1 - (e^{\gamma_j} - 1),$$

where $\gamma_j = \int_0^{+\infty} t |V_j(t)| dt$. Hence if $2 - e^{\gamma_j} > 0$, we deduce that $m_{j,+}(0, 0)$ is different from zero. This yields the conclusion since

$$m_{j,+}(0, 0) = f_{j,+}(0, 0) = \lim_{z \rightarrow 0} f_{j,+}(z, 0).$$

\square

Note that $V_j = 0$ is an exceptional case.

We now prove that $R_{j,2}(z)$ is continuous and uniformly bounded in B_κ if $\kappa > 0$ small enough (suggested by Remark 10 of [9]).

4.4 Lemma. *For all $j = 1, \dots, N$, there exists a positive constant C_j such that*

$$|R_{j,2}(z)| \leq C_j, \quad \forall z \in B_\kappa, \quad (4.41)$$

for $\kappa > 0$ small enough.

Proof. By Theorem 1 of [9], there exists $C_1 > 0$ such that

$$|T_j(z)| \leq C_1, \quad \forall z \in B_\kappa,$$

for $\kappa > 0$ small enough. Hence by (4.31) we deduce that (4.41) holds for all $|z| > \epsilon$, for any $\epsilon > 0$.

For z in the ball $|z| \leq \epsilon$, we distinguish the generic case to the exceptional one. In the generic case, by part V of Theorem 1 of [9], we know that

$$T_j(z) = \alpha_j z + o(z), \quad \text{for } z \rightarrow 0$$

with $\alpha_j \neq 0$ and again using (4.31) we deduce that (4.41) for $|z| \leq \epsilon$.

In the exceptional case, by (4.31) we may write

$$R_{j,2}(z) = \frac{T_j(z)}{2iz} \left(\int_0^{+\infty} (e^{2izy} - 1) V_j(y) m_{j,+}(z, y) dy + \int_0^{+\infty} V_j(y) (m_{j,+}(z, y) - m_{j,+}(0, y)) dy \right),$$

because $\int_0^{+\infty} V_j(t) m_{j,+}(0, t) dt = 0$. Therefore we obtain that

$$|R_{j,2}(z)| \leq C_1 \left(\left| \int_0^{+\infty} \frac{e^{2izy} - 1}{2iz} V_j(y) m_{j,+}(z, y) dy \right| + \left| \int_0^{+\infty} V_j(y) \frac{m_{j,+}(z, y) - m_{j,+}(0, y)}{2iz} dy \right| \right).$$

For the first term of this right hand side, due to (4.26) we can directly apply the dominated convergence theorem to conclude that

$$\int_0^{+\infty} \frac{e^{2izy} - 1}{2iz} V_j(y) m_{j,+}(z, y) dy \rightarrow \int_0^{+\infty} y V_j(y) m_{j,+}(0, y) dy \quad \text{as } z \rightarrow 0.$$

Since this limit is finite, we deduce that

$$\left| \int_0^{+\infty} \frac{e^{2izy} - 1}{2iz} V_j(y) m_{j,+}(z, y) dy \right| \leq C,$$

for $|z|$ small enough.

For the second term, we use the same argument. Namely since \tilde{V}_j belongs to $L^1_2(\mathbb{R})$, by Remark 3 of [9], the derivative $\dot{m}_{k,+}$ of $m_{k,+}$ with respect to k exists and is continuous on \mathbb{C}^+ . Moreover by Lemma 2.1 of [18], there exists $C_2 > 0$ such that

$$|\dot{m}_{k,+}(z, y)| \leq C_2, \quad \forall x \geq 0. \quad (4.42)$$

Consequently by using the mean value theorem we have

$$\frac{m_{j,+}(z, y) - m_{j,+}(0, y)}{2iz} = \dot{m}_{k,+}(\theta z, y),$$

for some $\theta \in (0, 1)$ and therefore

$$\left| \frac{m_{j,+}(z, y) - m_{j,+}(0, y)}{2iz} \right| \leq C_2, \quad \forall x \geq 0.$$

The application of dominated convergence theorem yields

$$\int_0^{+\infty} V_j(y) \frac{m_{j,+}(z, y) - m_{j,+}(0, y)}{2iz} dy \rightarrow \int_0^{+\infty} V_j(y) \dot{m}_{j,+}(0, y) dy \quad \text{as } z \rightarrow 0.$$

The conclusion follows since this right-hand side is finite. \square

We are now ready to give the different families of generalized eigenfunctions of H .

4.5 Lemma. *Under the assumption (4.36), then for all $z \in B_\kappa$, $z \neq 0$ and all $j \in \{1, \dots, N\}$, there exist two generalized eigenfunctions $F_{z^2}^{\pm, j} : \mathcal{R} \rightarrow C$ of H defined by*

$$F_{z^2}^{\pm, j}(x) := F_{z^2, k}^{\pm, j}(x) \quad \forall x \in \overline{R_k},$$

where $F_{z^2, k}^{\pm, j}$ is in the form

$$\begin{cases} F_{z^2, j}^{\pm, j}(x) &= c_{j, \pm, 1}(z) f_{j, \pm}(z, x) + c_{j, \pm, 2}(z) f_{j, \mp}(z, x), \\ F_{z^2, k}^{\pm, j}(x) &= d_{j, k, \pm}(z) f_{k, \mp}(z, x), \forall k \neq j, \end{cases} \quad (4.43)$$

and $c_{j, \pm, 1}(z)$, $c_{j, \pm, 2}(z)$ and $d_{j, k, \pm}(z)$ are given by (modulo N)

$$\begin{aligned} c_{j, \pm, 1}(z) &= \frac{f_{j+1, \mp}(z, 0)}{W_{j, \pm}(z)} \left(f'_{j, \mp}(z, 0) + f_{j, \mp}(z, 0) \sum_{k \neq j} \frac{f'_{k, \mp}(z, 0)}{f_{k, \mp}(z, 0)} \right), \\ c_{j, \pm, 2}(z) &= -\frac{f_{j+1, \mp}(z, 0)}{W_{j, \pm}(z)} \left(f'_{j, \pm}(z, 0) + f_{j, \pm}(z, 0) \sum_{k \neq j} \frac{f'_{k, \mp}(z, 0)}{f_{k, \mp}(z, 0)} \right), \\ d_{j, k, \pm}(z) &= \frac{f_{j+1, \mp}(z, 0)}{f_{k, \mp}(z, 0)}, \forall k \neq j, \end{aligned}$$

$W_{j, \pm}(z)$ is the Wronskian relatively to $f_{j, \pm}$, namely

$$W_{j, \pm}(z) = f_{j, \pm}(z, x) f'_{j, \mp}(z, x) - f_{j, \mp}(z, x) f'_{j, \pm}(z, x),$$

that is constant in x and different from 0 (since $z \neq 0$).

Proof. We look for generalized eigenfunctions in the form (4.43), the constants $c_{j, \pm, 1}(z)$, $c_{j, \pm, 2}(z)$ and $d_{j, k, \pm}(z)$ will be fixed below in order to garanties the continuity of $F_{z^2}^{\pm, j}$ at 0 and the Kirchoff law. This will show that $F_{z^2}^{\pm, j}$ are generalized eigenfunctions of H since $F_{z^2, k}^{\pm, j}$ satisfies

$$-\frac{d^2}{dx^2} F_{z^2, k}^{\pm, j}(x) + \tilde{V}_j(x) F_{z^2, k}^{\pm, j}(z, x) = z^2 F_{z^2, k}^{\pm, j} \quad \text{on } R_k.$$

Since each branch j plays the same rule, we can take $j = 1$ and write $c_{1, \pm, 1}(z) = c_1$, $c_{1, \pm, 2}(z) = c_2$ and $d_{1, k, \pm}(z) = d_k$. The continuity at 0 is equivalent to

$$c_1 f_{1, \pm}(z, 0) + c_2 f_{1, \mp}(z, 0) = d_k f_{k, \mp}(z, 0) \quad \forall k \neq 1,$$

while the Kirchoff law is equivalent to

$$c_1 f'_{1, \pm}(z, 0) + c_2 f'_{1, \mp}(z, 0) + \sum_{k=2}^N d_k f'_{k, \mp}(z, 0) = 0.$$

Since by Lemma 4.1 $f_{k, \mp}(z, 0)$ is different from 0, we will get

$$d_k = \frac{d_2 f_{2, \mp}(z, 0)}{f_{k, \mp}(z, 0)}, \forall k \neq 1,$$

and the continuity and the Kirchoff law reduce to

$$\begin{cases} c_1 f_{1, \pm}(z, 0) + c_2 f_{1, \mp}(z, 0) = d_2 f_{2, \mp}(z, 0), \\ c_1 f'_{1, \pm}(z, 0) + c_2 f'_{1, \mp}(z, 0) = -d_2 f_{2, \mp}(z, 0) \sum_{k=2}^N \frac{f'_{k, \mp}(z, 0)}{f_{k, \mp}(z, 0)}. \end{cases}$$

This 2×2 linear system in c_1 and c_2 has a unique solution since its determinant is exactly $W_{1, \pm}(z)$. The resolution of this system leads to the conclusion with the choice $d_2 = 1$. \square

4.6 Remark. The choice (4.43) was guided by the simple case when $N = 2$ and $V_k = 0, k = 1, 2$. In that case, we recover the standard generalized eigenfunctions, namely

$$F_{z^2,1}^{\pm,1}(x) = e^{\pm izx}, \forall x > 0,$$

as well as

$$F_{z^2,2}^{\pm,1}(x) = e^{\mp izx}, \forall x > 0.$$

According to Lemma 4.1, we see that

$$c_{j,+1}(z) = -\frac{izN}{W_{j,+}(z)},$$

which is always different from 0 if $z \in \mathbb{C}^+, z \neq 0$, while

$$c_{j,-1}(z) = \frac{izf_{j+1,+}(z,0)}{W_{j,-}(z)} \sum_{k=1}^N \frac{1 - R_{k,2}(z)}{1 + R_{k,2}(z)},$$

is not clearly different from zero. This is investigated in the next Lemma

4.7 Lemma. *Under the assumption (4.36), there exists $\kappa > 0$ small enough such that*

$$s(z) := \sum_{k=1}^N \frac{1 - R_{k,2}(z)}{1 + R_{k,2}(z)},$$

satisfies

$$|s(z)| \geq C, \forall z \in B_\kappa, \quad (4.44)$$

for some $C > 0$.

Proof. Clearly s is continuous on $B_\kappa \setminus \{0\}$ for κ small enough, hence we first analyze the behaviour of s near $z = 0$.

For $z \in B_\kappa \setminus \{0\}$ and $k \in \{1, \dots, N\}$, we write

$$s_k(z) := \frac{1 - R_{k,2}(z)}{1 + R_{k,2}(z)} = \frac{1 - R_{k,2}(z)}{T_k(z)} \frac{T_k(z)}{1 + R_{k,2}(z)}.$$

The absolute value of the second factor is uniformly bounded from below on B_κ thanks to Lemmas 4.1 and 4.2.

For the first factor, we distinguish between the generic and the exceptional case: In the exceptional case,

$$|T_k(z)| \geq c_k, \forall z \in B_\kappa,$$

for some $c_k > 0$ (and κ small enough) and therefore s_k is continuous on B_κ .

In the generic case, using (4.30) and (4.31), we may write

$$\frac{1 - R_{k,2}(z)}{T_k(z)} = 1 - \int_0^{+\infty} \frac{1 + e^{2izy}}{2iz} \tilde{V}_k(y) m_{k,+}(z, y) dy \quad \forall z \in \mathbb{C}^+, z \neq 0.$$

As underlined before, the derivative $\dot{m}_{k,+}$ of $m_{k,+}$ with respect to k exists, is continuous on \mathbb{C}^+ and satisfies (4.42). Accordingly, using the mean value theorem and the dominated convergence theorem, we get for all $z \neq 0$ small enough

$$\frac{1 - R_{k,2}(z)}{T_k(z)} = 1 - \frac{\nu_k}{iz} + r_k(z),$$

where r_k is a continuous function at $z = 0$ and $\nu_k = \int_0^{+\infty} V_k(t) m_{k,+}(0, t) dt$ (that is different from zero because we are in the generic case).

In the same manner we can refine (4.39) and prove that

$$\frac{1 + R_{k,2}(z)}{T_k(z)} = \gamma_k + z r_k^{(1)}(z),$$

where $r^{(1)}$ is a continuous function at $z = 0$ and $\gamma_k = 1 + \int_0^{+\infty} tV_k(t)m_{k,+}(0,t) dt$ that by hypothesis is a real number different from 0. Consequently for z small enough we will get

$$\frac{T_k(z)}{1 + R_{k,2}(z)} = \gamma_k^{-1} + zr_k^{(2)}(z), \quad (4.45)$$

where $r^{(2)}$ is a continuous function at $z = 0$.

The two previous expansions show that for all $z \neq 0$ small enough

$$s_k(z) = -\frac{\nu_k}{i\gamma_k z} + r_k^{(3)}(z),$$

where $r^{(3)}$ is a continuous function at $z = 0$.

In summary, we have obtained that for all $z \neq 0$ small enough

$$s(z) = -\frac{1}{iz} \sum_{k \text{ generic}} \frac{\nu_k}{\gamma_k} + r(z),$$

where r is a continuous function at $z = 0$.

Now we can distinguish two cases:

- i) If $\sum_{k \text{ generic}} \frac{\nu_k}{\gamma_k} = 0$, then s is continuous at $z = 0$, and therefore s is continuous on B_κ .
- ii) If $K := \sum_{k \text{ generic}} \frac{\nu_k}{\gamma_k} \neq 0$, then s blows up at $z = 0$ and therefore there exists δ_0 small enough such that

$$|s(z)| \geq \frac{K}{2|z|}, \forall |z| < \delta_0. \quad (4.46)$$

Now for $|z|$ large, by (4.38) we have

$$\lim_{|z| \rightarrow +\infty} s_k(z) = 1,$$

hence there exists R_0 large enough such that

$$\Re s(z) \geq \frac{N}{2}, \forall z \in B_\kappa : |z| > R_0. \quad (4.47)$$

For small value of $|z|$, we first restrict ourselves on the real line. First we notice that

$$\Re s_k(z) = \Re \frac{1 - R_{k,2}(z)}{1 + R_{k,2}(z)} = \frac{1 - |R_{k,2}(z)|^2}{|1 + R_{k,2}(z)|^2}.$$

But according to parts II and V of Theorem 1 of [9],

$$|R_{k,2}(z)| < 1, \quad \forall z \in \mathbb{R}, z \neq 0,$$

and therefore

$$\Re s_k(z) > 0, \quad \forall z \in \mathbb{R}, z \neq 0.$$

Now thanks to (4.39) and to the relation

$$1 - |R_{k,2}(z)|^2 = |T_k(z)|^2,$$

valid for all real numbers z , we deduce that

$$\lim_{\substack{z \rightarrow 0 \\ z \in \mathbb{R}}} \frac{1 - |R_{k,2}(z)|^2}{|1 + R_{k,2}(z)|^2} = \frac{1}{\gamma_k^2},$$

where $\gamma_k = 1 + \int_0^{+\infty} tV_j(t)m_{j,+}(0,t) dt$ that by hypothesis is a real number different from 0.

This shows that

$$\lim_{\substack{z \rightarrow 0 \\ z \in \mathbb{R}}} \Re s(z) = \sum_{k=1}^N \frac{1}{\gamma_k^2},$$

and consequently as $\Re s$ is a continuous function on \mathbb{R} that is different from zero for all real numbers, due to (4.47), it satisfies

$$\Re s(z) \geq C, \forall z \in \mathbb{R}, \quad (4.48)$$

for some $C > 0$.

In the first case mentioned before, namely if $K = 0$, then by the uniform continuity of $\Re s$ on the compact set $B_\kappa \cap \{z_1 \in \mathbb{C} : 0 \leq z_1 \leq R_0\}$, where R_0 is the parameter introduced above, we deduce that

$$\Re s(z) \geq C/2, \forall z \in B_{\kappa'} \cap \{z_1 \in \mathbb{C} : |z_1| \leq R_0\}, \quad (4.49)$$

if κ' is chosen small enough. In that case the conclusion directly follows from (4.47) and (4.49).

In the case when $K \neq 0$, we use the uniform continuity of $\Re s$ on the compact set $B_\kappa \cap \{z_1 \in \mathbb{C} : \frac{\delta_0}{2} \leq |z_1| \leq R_0\}$ (where R_0, δ_0 are the parameter introduced above), and (4.48) to conclude that

$$\Re s(z) \geq C/2, \forall z \in B_{\kappa'} \cap \{z_1 \in \mathbb{C} : \frac{\delta_0}{2} \leq |z_1| \leq R_0\}, \quad (4.50)$$

if κ' is chosen small enough.

In this second case the conclusion follows from (4.46), (4.47) and (4.49). \square

4.8 Corollary. *Under the assumption (4.36), for $\kappa > 0$ small enough there exist two positive constants c_1, c_2 such that*

$$|c_{j,-,1}(z)W_{j,-}(z)| \geq c_1|z|, \quad \forall z \in B_\kappa, \quad (4.51)$$

$$|c_{j,-,2}(z)| \leq c_2|s(z)|, \quad \forall z \in B_\kappa. \quad (4.52)$$

Proof. As

$$c_{j,-,1}(z) = \frac{izf_{j+1,+}(z,0)}{W_{j,-}(z)}s(z),$$

by the previous Lemma and Lemma 4.2, we deduce that (4.51) holds.

By its definition and Lemma 4.1, we may write

$$c_{j,-,2}(z) = iz \frac{f_{j+1,+}(z,0)}{W_{j,-}(z)} \left(1 + \sum_{k \neq j} \frac{1 - R_k(z)}{1 - R_k(z)} \right),$$

hence thanks to the definition of $s(z)$, we obtain

$$c_{j,-,2}(z) = iz \frac{f_{j+1,+}(z,0)}{W_{j,-}(z)} \left(\frac{2R_{j,2}(z)}{1 + R_{j,2}(z)} + s(z) \right).$$

Now recalling that

$$W_{j,-}(z) = -W_{j,+}(z) = -\frac{2iz}{T_j(z)},$$

we can write

$$c_{j,-,2}(z) = -\frac{f_{j+1,+}(z,0)}{2} \left(\frac{2R_{j,2}(z)T_j(z)}{1 + R_{j,2}(z)} + s(z)T_j(z) \right). \quad (4.53)$$

By Lemmas 4.1, 4.2, 4.4 and 4.7 we deduce that there exists $C_1 > 0$ such that

$$|c_{j,-,2}(z)| \leq C_1(1 + |s(z)|) \leq \left(\frac{C_1}{C} + C_1 \right) |s(z)|,$$

with the constant C from (4.44). \square

4.9 Corollary. *Under the assumption (4.36), and if $V_k \in L_\gamma^1(0, \infty)$ with $\gamma > 5/2$, for all $k = 1, \dots, N$, then for all $R > 0$, s^{-1} belongs to $H^1(-R, R)$.*

Proof. With the notation from the previous Lemma, we see that r_k is given by

$$r_k(z) = \int_0^{+\infty} \frac{V_k(y)}{2iz} (2m_{k,+}(0, y) - (1 + e^{2izy})m_{k,+}(z, y)) dy,$$

and is continuous on \mathbb{R} . Moreover for $z \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ we easily see that r_k is differentiable at z and that

$$\dot{r}_k(z) = - \int_0^{+\infty} \frac{V_k(y)}{2iz^2} (2m_{k,+}(0, y) - g_{k,+}(z, y) + z\dot{g}_{k,+}(z, y)) dy.$$

where for shortness we have set

$$g_{k,+}(z, y) := (1 + e^{2izy})m_{k,+}(z, y).$$

But the mean value theorem implies that

$$g_{k,+}(z, y) = 2m_{k,+}(0, y) + z\dot{g}_{k,+}(\theta z, y),$$

for some $\theta \in (0, 1)$ and therefore

$$\dot{r}_k(z) = \int_0^{+\infty} \frac{V_k(y)}{2iz} (\dot{g}_{k,+}(\theta z, y) - \dot{g}_{k,+}(z, y)) dy, \quad \forall z \in \mathbb{R}^*.$$

As

$$\dot{g}_{k,+}(z, y) = 2iy e^{2izy} m_{k,+}(z, y) + (1 + e^{2izy})\dot{m}_{k,+}(z, y),$$

the previous identity can be equivalently written

$$\begin{aligned} \dot{r}_k(z) &= \int_0^{+\infty} V_k(y) \left(y m_{k,+}(\theta z, y) \frac{e^{2i\theta zy} - e^{2izy}}{z} \right. \\ &\quad + y e^{2izy} \frac{m_{k,+}(\theta z, y) - m_{k,+}(z, y)}{z} \\ &\quad + \frac{e^{2i\theta zy} - e^{2izy}}{z} \dot{m}_{k,+}(z, y) \\ &\quad \left. + (1 + e^{2izy}) \frac{\dot{m}_{k,+}(\theta z, y) - \dot{m}_{k,+}(z, y)}{z} \right) dy, \quad \forall z \in \mathbb{R}^*. \end{aligned}$$

Again by the mean value theorem we get

$$\begin{aligned} \dot{r}_k(z) &= \int_0^{+\infty} V_k(y) \left(2iy^2 m_{k,+}(\theta z, y) e^{2i\theta' zy} (\theta - 1) \right. \\ &\quad + y e^{2izy} \dot{m}_{k,+}(\theta'' z, y) (\theta - 1) \\ &\quad + 2iy e^{2i\theta' zy} (\theta - 1) \dot{m}_{k,+}(z, y) \\ &\quad \left. + (1 + e^{2izy}) \frac{\dot{m}_{k,+}(\theta z, y) - \dot{m}_{k,+}(z, y)}{z} \right) dy, \quad \forall z \in \mathbb{R}^*, \end{aligned}$$

for some $\theta', \theta'' \in (\theta, 1)$. Note that we cannot apply the mean value theorem to the last term since $\dot{m}_{k,+}$ is not differentiable. But according to Lemma 2.2 of [18] we have

$$|\dot{m}_{k,+}(z, y) - \dot{m}_{k,+}(0, y)| \leq C|z|^{\gamma-2}, \quad \forall y \geq 0, \quad (4.54)$$

for some $C > 0$ independent of z and y . This estimate, (4.26) and (4.42) lead to

$$|\dot{r}_k(z)| \leq C \int_0^{+\infty} |V_k(y)| (y^2 + y + |z|^{\gamma-2}) dy, \quad \forall z \in \mathbb{R}^*.$$

for some $C > 0$. Hence according to our hypothesis on V_k , we get

$$|\dot{r}_k(z)| \leq C_1(1 + |z|^{\gamma-3}), \quad \forall z \in \mathbb{R}^*,$$

for some $C_1 > 0$.

This estimate and the continuity of r_k imply that r_k belong to $H^1(-R, R)$ for any $R > 0$ due to the hypothesis $\gamma > 5/2$.

In the same way we need to precise the splitting (4.45) on the real line (actually near 0). For that purpose, we consider

$$g_k(z) := \frac{m_{k,+}(z, 0) - m_{k,+}(0, 0)}{z}, \forall z \in \mathbb{R},$$

and show that g_k belongs to $H^1(-R, R)$ for any $R > 0$. First g_k is continuous at 0 because $m_{k,+}(z, 0)$ is in $C^1(\mathbb{R})$. Second by Leibniz's rule we have

$$\dot{g}_k(z) = \frac{\dot{m}_{k,+}(z, 0)z - (m_{k,+}(z, 0) - m_{k,+}(0, 0))}{z^2}$$

and therefore by the mean value theorem we get

$$\dot{g}_k(z) = \frac{\dot{m}_{k,+}(z, 0) - \dot{m}_{k,+}(\theta z, 0)}{z},$$

for some $\theta \in (0, 1)$ and we conclude by (4.54).

But we see that

$$\frac{(m_{k,+}(z, 0))^{-1}}{z} = \frac{(m_{k,+}(0, 0))^{-1}}{z} + h_k(z) = \frac{1}{\gamma_k z} + h_k(z)$$

with

$$h_k(z) = \frac{m_{k,+}(z, 0) - m_{k,+}(0, 0)}{zm_{k,+}(z, 0)m_{k,+}(0, 0)} = \frac{g_k(z)}{m_{k,+}(z, 0)m_{k,+}(0, 0)}.$$

According to the previous considerations, g_k belongs to $H^1(-R, R)$, for any $R > 0$ and since $m_{k,+}(\cdot, 0)$ belongs to $C^1(\mathbb{R})$ and is uniformly bounded from below (due to Lemmas 4.1 and 4.2), $\frac{1}{m_{k,+}(\cdot, 0)}$ is also in $C^1(\mathbb{R})$. Therefore h_k also belongs to $H^1(-R, R)$, for any $R > 0$.

Coming back to s , recalling that

$$s(z) = \sum_{k=1}^N (1 + \frac{i\nu_k}{z} + r_k(z))(m_{k,+}(z, 0))^{-1},$$

we have finally shown that

$$s(z) = i\frac{K}{z} + r_s(z),$$

where r_s belongs to $H^1(-R, R)$, for any $R > 0$.

Now we distinguish the case $K = 0$ to the other one: In the first case, we have that $s = r_s$ belongs to $H^1(-R, R)$, for any $R > 0$ and since s is uniformly bounded from below by the previous Lemma, we deduce that $\frac{1}{s}$ belongs to $H^1(-R, R)$, for any $R > 0$.

If $K \neq 0$, then

$$\frac{1}{s(z)} = \frac{z}{iK + zr_s(z)},$$

that is a continuous function in \mathbb{R} and moreover for $z \in \mathbb{R}^*$, we have after elementary calculations

$$\frac{d}{dz} \frac{1}{s}(z) = \frac{iK + z^2 \dot{r}_s(z)}{(iK + zr_s(z))^2}.$$

Since this right-hand side is in $L^2(-R, R)$, for any $R > 0$ (because the denominator is different from zero near $z = 0$, while by the previous Lemma, for any $z \in \mathbb{R}^*$ $s(z) \geq C$ is equivalent to $|iK + zr_s(z)| \geq C|z|$), we still conclude that $\frac{1}{s}$ belongs to $H^1(-R, R)$, for any $R > 0$. \square

4.10 Corollary. *Under the assumption (4.36), and if $V_k \in L^1_\gamma(0, +\infty)$ with $\gamma > 5/2$, then the function*

$$\mathbb{R} \rightarrow C : z \rightarrow \frac{c_{j,-,2}(z)}{f_{j+1,+}(z, 0)s(z)},$$

belongs to $H^1(-R, R)$ for all $R > 0$.

Proof. By (4.53), we see that

$$\frac{c_{j,-,2}(z)}{f_{j+,+}(z,0)s(z)} = -\frac{1}{2} \left(\frac{2R_{j,2}(z)T_j(z)}{(1+R_{j,2}(z))s(z)} + T_j(z) \right) = -\frac{1}{2} \left(\frac{2R_{j,2}(z)}{f_{j+,+}(z,0)s(z)} + T_j(z) \right).$$

But according to Remark 10 of [9], T_j is analytic in a neighbourhood of the real line, hence it is at least in $C^1(\mathbb{R})$. On the other hand $f_{j+,+}(z,0) = m_{j+,+}(z,0)$ is $C^1(\mathbb{R})$ due to Remark 3 of [9], hence $\frac{1}{f_{j+,+}(z,0)}$ has the same property due to Lemma 4.2. Finally the identity (4.35) of Lemma 4.1 yields

$$R_{j,2}(z) = f_{j+,+}(z,0)T_j(z) - 1,$$

hence it also belongs to $C^1(\mathbb{R})$.

The conclusion follows from the previous Corollary and these regularity properties (the product of a C^1 function with a H^1 function is still in H^1). \square

4.11 Definition (Kernel of the resolvent). *Let the assumption (4.36) be satisfied, then for all $z \in B_\kappa, z \neq 0$, all $j \in \{1, \dots, N\}$, and all $x \in R_j$, we define (modulo N)*

$$K(x, x', z^2) = \begin{cases} \frac{1}{W_j(z)} F_{z^2,j}^{-,j}(x) F_{z^2,j}^{-,j+1}(x'), & \text{for } x' \in R_j, x' > x, \\ \frac{1}{W_j(z)} F_{z^2,j}^{-,j+1}(x) F_{z^2,j}^{-,j}(x'), & \text{for } x' \in R_j, x' < x, \\ \frac{1}{W_j(z)} F_{z^2,j}^{-,j+1}(x) F_{z^2,k}^{-,j}(x'), & \text{for } x' \in R_k, k \neq j, \end{cases}$$

where $W_j(z) = c_{j,-,1}(z)d_{j+1,j,-}(z)W_{j,-}(z)$.

4.12 Theorem. *Let the assumption (4.36) be satisfied and let $f \in \mathcal{H}$. Then, for $x \in \mathcal{R}$ and $z \in B_\kappa$ such that $\Im z > 0$, we have*

$$[R(z^2, A)f](x) = \int_{\mathcal{R}} K(x, x', z^2) f(x') dx'. \quad (4.55)$$

Proof. Fix $j \in \{1, \dots, N\}$, and z as in the statement. Then we notice that the Wronskian $W_j(z)$ between $F_{z^2,j}^{-,j}$ and $F_{z^2,j}^{-,j+1}$ is different from zero, namely by Lemma 4.5 we have

$$\begin{aligned} W_j(z) &= [F_{z^2,j}^{-,j}, F_{z^2,j}^{-,j+1}](x) \\ &= F_{z^2,j}^{-,j}(x) \left(F_{z^2,j}^{-,j+1} \right)'(x) - \left(F_{z^2,j}^{-,j} \right)'(x) F_{z^2,j}^{-,j+1}(x) \\ &= (c_{j,-,1}(z)f'_{j,-}(z, x) + c_{j,-,2}(z)f'_{j,+}(z, x))d_{j+1,j,-}(z)f_{j,+}(z, x) \\ &\quad - (c_{j,-,1}(z)f_{j,-}(z, x) + c_{j,-,2}(z)f_{j,+}(z, x))d_{j+1,j,-}(z)f'_{j,+}(z, x) \\ &= c_{j,-,1}(z)d_{j+1,j,-}(z)W_{j,-}(z) \end{aligned}$$

Hence by Lemma 4.2 and Corollary 4.8 this Wronskian is different from zero.

Consequently the same arguments than in Proposition 3.2 of [3] show that (4.55) holds. The main ingredient is that we can apply the dominated convergence theorem because the generalized eigenfunction $F_{z^2,k}^{-,j}$ is in $L^2(R_k)$ if $j \neq k$. \square

4.13 Remark. The choice of the kernel comes from this Theorem because $F_{z^2,k}^{+,j}$ is not in $L^2(R_k)$ if $j \neq k$.

Here and below the complex square root is chosen in such a way that $\sqrt{r \cdot e^{i\phi}} = \sqrt{r}e^{i\phi/2}$ with $r > 0$ and $\phi \in [-\pi, \pi)$. Accordingly for any positive real number λ and any $\varepsilon > 0$, we will define

$$z_\varepsilon = \sqrt{\lambda + i\varepsilon}$$

that will be in \mathbb{C}^+ .

4.14 Theorem (Limiting absorption principle). *Let the assumption (4.36) be satisfied. Let $\delta > 0$ be fixed. Then for all real numbers $\lambda > 0$, $0 < \varepsilon < \delta$ and $(x, x') \in \mathcal{R}^2$ we have*

$$1. \lim_{\substack{\alpha \rightarrow 0 \\ \alpha > 0}} K(x, x', z_\alpha^2) = K(x, x', \lambda),$$

$$2. |K(x, x', z_\varepsilon^2)| \leq \frac{C}{\sqrt{\lambda}} e^{\gamma(x+x')}, \text{ where } 0 < \gamma < \max\{1, \delta\}.$$

Proof. The first part of the Theorem is direct since $\lambda + i\alpha$ tends to λ as $\alpha > 0$ tends to 0 and consequently

$$\sqrt{\lambda + i\alpha} \rightarrow \sqrt{\lambda},$$

as $\alpha > 0$ tends to 0. We further use the fact that the functions $f_{j,\pm}(\cdot, x)$ and $f'_{j,\pm}(\cdot, x)$ are continuous in \mathbb{C}^+ for any fixed $x \in \mathbb{R}$.

For the second part of the Theorem, we first use the estimates (4.28) and (4.29), this last one implying

$$|f_{j,-}(z_\varepsilon, x)| \leq C(1+x)e^{\Im z_\varepsilon x} \leq C(1+x)e^{\max\{1, \delta\}x}, \quad \forall x \in [0, +\infty), \quad (4.56)$$

where we have used the property

$$\Im z_\varepsilon = |\Im \sqrt{\lambda + i\varepsilon}| \leq \max\{1, \Im(\lambda + i\varepsilon)\} = \max\{1, \varepsilon\}.$$

Notice that by the definition $W_j(z) = c_{j,-,1}(z)d_{j+1,j,-}(z)W_{j,-}(z)$ and by Lemma 4.2 and Corollary 4.8, we get

$$|W_j(z)| \geq C|z|, \quad (4.57)$$

for some $C > 0$.

Now we distinguish between the following three cases:

1. If $x, x' \in R_j$ with $x' > x$, then

$$\begin{aligned} K(x, x', z_\varepsilon^2) &= \frac{1}{W_j(z_\varepsilon)} F_{z_\varepsilon^2, j}^{-,j}(x) F_{z_\varepsilon^2, j}^{-,j+1}(x') \\ &= \frac{1}{W_j(z_\varepsilon)} \left(c_{j,-,1}(z_\varepsilon) f_{j,-}(z_\varepsilon, x) + c_{j,-,2}(z_\varepsilon) f_{j,+}(z_\varepsilon, x) \right) d_{j+1,j,-}(z_\varepsilon) f_{j,+}(z_\varepsilon, x') \\ &= \frac{1}{W_{j,-}(z_\varepsilon)} f_{j,-}(z_\varepsilon, x) f_{j,+}(z_\varepsilon, x') + \frac{c_{j,-,2}(z_\varepsilon)}{iz_\varepsilon f_{j+1,+}(z_\varepsilon, 0) s(z_\varepsilon)} f_{j,+}(z_\varepsilon, x) f_{j,+}(z_\varepsilon, x'). \end{aligned}$$

As there exists $c > 0$ such that

$$|W_{j,-}(z)| \geq c|z|, \quad \forall z \in \mathbb{C}^+,$$

by Lemma 4.2 and Corollary 4.8, we obtain

$$|K(x, x', z_\varepsilon)| \leq \frac{C}{|z_\varepsilon|} (|f_{j,-}(z_\varepsilon, x)| + |f_{j,+}(z_\varepsilon, x)|) |f_{j,+}(z_\varepsilon, x')|.$$

The estimates (4.28) and (4.56) then yields

$$|K(x, x', z_\varepsilon^2)| \leq \frac{C}{|z_\varepsilon|} (1 + (1+x)e^{\max\{1, \delta\}x}). \quad (4.58)$$

2. If $x, x' \in R_j$ with $x' > x$, then

$$K(x, x', z_\varepsilon^2) = \frac{1}{W_j(z_\varepsilon)} F_{z_\varepsilon^2, j}^{-,j+1}(x) F_{z_\varepsilon^2, j}^{-,j}(x'),$$

and the above arguments (by simply exchanging the role of x and x') yields

$$|K(x, x', z_\varepsilon)| \leq \frac{C}{|z_\varepsilon|} (1 + (1+x')e^{\max\{1, \delta\}x'}). \quad (4.59)$$

3. If $x \in R_j$ and $x' \in R_k$ with $k \neq j$, we have

$$\begin{aligned} K(x, x', z_\varepsilon^2) &= \frac{1}{W_j(z_\varepsilon)} F_{z_\varepsilon^2, j}^{-,j+1}(x) F_{z_\varepsilon^2, k}^{-,j}(x') \\ &= \frac{1}{W_j(z_\varepsilon)} d_{j+1,j,-}(z_\varepsilon) f_{j,+}(z_\varepsilon, x) d_{j+1,k,-}(z_\varepsilon) f_{k,+}(z_\varepsilon, x'). \end{aligned}$$

Hence by Lemma 4.2 and the estimates (4.28) and (4.57), we obtain

$$|K(x, x', z_\varepsilon^2)| \leq \frac{C}{|z_\varepsilon|}. \quad (4.60)$$

The estimates (4.58), (4.59) and (4.60) imply the conclusion since $|z_\varepsilon| > \sqrt{\lambda}$. \square

4.15 Theorem. Take $f \in \mathcal{H}$ with a compact support and let $0 \leq a < b < +\infty$. Then for any continuous scalar function h defined on the real line and for all $x \in R_j$, we have

$$(h(H)E(a,b)f)(x) = - \int_{(a,b)} h(\lambda) \sum_{k=1}^N \int_{R_k} f(x') \Im K(x, x', \lambda) dx' d\lambda,$$

where E is the resolution of the identity of H .

Proof. The proof is similar to the one of Lemma 3.13 of [1] (see also Proposition 4.5 of [4]) and is therefore omitted. The main ingredients are the use of Stone's formula, Theorem 4.12 and the limiting absorption principle Theorem 4.14 (that allows to apply the dominated convergence theorem). \square

5 Proof of Theorem 1.1

5.1 High energy estimate

It is clear that for $V \in L^1(\mathcal{R})$, H is essentially self-adjoint on the domain

$$\left\{ f = (f_k) \in L^2(\mathcal{R}); f, f' \text{ a.c. and } -f'' + Vf \in L^2(\mathcal{R}); f_j(0) = f_k(0), \forall j, k = 1, \dots, N, \right. \\ \left. \sum_{k=1}^N \frac{df_k}{dx}(0) = 0 \right\}$$

so that e^{itH} is unitary. Hence the dispersive estimate is to be understood as the statement

$$\|e^{itH} P_{ac} f\|_{\infty} \leq |t|^{-1/2} \|f\|_1, \forall f \in L^1(\mathcal{R}) \cap L^2(\mathcal{R}).$$

Which then extends to all of $L^1(\mathcal{R})$. We start with the high energy part of the argument.

5.1 Lemma. Let $\lambda_0 = \frac{4(N-1)^2 \|V\|^2}{N^2}$ and suppose χ is a smooth cut-off such that $\chi(\lambda) = 0$ for $\lambda \leq \lambda_0$ and $\chi(\lambda) = 1$ for $\lambda \geq 2\lambda_0$. Then there exists $C > 0$ such that

$$\|e^{itH} \chi(H)\|_{1,\infty} \leq C |t|^{-1/2}, \forall t \neq 0.$$

Proof. In the limit $\varepsilon \rightarrow 0^+$ the resolvent $R_0(\lambda + i\varepsilon) = \left(-\frac{d^2}{dx^2} - (\lambda + i\varepsilon)\right)^{-1}$, according to [3] has the kernel, for $x \in \overline{R_j}, j = 1, \dots, N$,

$$K_0(x, x', \lambda \pm i0) = \frac{\pm i}{N\sqrt{\lambda}} \begin{cases} e^{\pm i(x+x')\sqrt{\lambda}}, x' \in \overline{R_k}, k \neq j, \\ \frac{N}{2} e^{\pm i|x-x'|\sqrt{\lambda}} + (1 - \frac{N}{2}) e^{\pm i(x+x')\sqrt{\lambda}}, \forall x' \in \overline{R_j}, \end{cases}$$

and therefore

$$|K(x, x', \lambda \pm i0)| \leq \frac{N-1}{N\sqrt{\lambda}}, \forall (x, x') \in \mathcal{R}^2. \quad (5.61)$$

Then, because of the decay of this kernel in λ , the resolvent $R_V(\lambda + i\varepsilon) = (H - (\lambda + i\varepsilon))^{-1}$ can be expanded into the Born series

$$R_V(\lambda \pm i0) = \sum_{k \geq 0} R_0(\lambda \pm i0) (-V R_0(\lambda \pm i0))^k. \quad (5.62)$$

More precisely, due to (5.61) one has

$$\|R_0(\lambda \pm i0)\|_{1,\infty} \leq (N-1)(N\sqrt{\lambda})^{-1},$$

and therefore due the assumption on V , one gets

$$\|V R_0(\lambda \pm i0)\|_{1,1} \leq \frac{N-1}{N\sqrt{\lambda}} \|V\|_1.$$

so that (5.62) converges provided $\lambda > \lambda_0 = \frac{4(N-1)^2 \|V\|_1^2}{N^2}$ in the following weak sense :

$$\langle R_V(\lambda \pm i0)f, g \rangle = \sum_{k \geq 0} \langle R_0(\lambda \pm i0)(-VR_0(\lambda \pm i0))^k f, g \rangle$$

for any $f, g \in L^1(\mathcal{R})$. For such functions it is a standard fact that

$$R_V(\lambda - i0)g \in L^\infty(\mathcal{R})$$

provided $\lambda > 0$. Therefore, the error term in any finite Born expansion. i.e., $R_V(\lambda + i0)(VR_0(\lambda + i0))^k$, tends to zero weakly as $k \rightarrow \infty$ provided $\lambda > \lambda_0$ since

$$\begin{aligned} |\langle R_V(\lambda + i0)(VR_0(\lambda + i0))^k f, g \rangle| &\leq \| (VR_0(\lambda + i0))^k f \|_1 \|R_V(\lambda - i0)g\|_\infty \\ &\leq \left(\frac{N-1}{N\sqrt{\lambda}} \right)^k \|V\|_1^k \|f\|_1 \|R_V(\lambda - i0)g\|_\infty. \end{aligned}$$

We introduce now a truncated version χ_L of the cut-off $\chi : \chi_L = \chi(\lambda)\phi(\frac{\lambda}{L})$, where ϕ is smooth,

$$\phi(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \leq 1 \\ 0 & \text{if } |\lambda| \geq 2, \end{cases}$$

and $L \geq 1$.

We need to show that

$$\sup_{L \geq 1} |\langle e^{itH} \chi_L(H)f, g \rangle| \leq C|t|^{-1/2} \|f\|_1 \|g\|_1$$

for any $f, g \in \mathcal{C}_0^\infty(\mathcal{R})$. The absolutely continuous part of the spectral measure of H , which we denote by $E_{ac}(d\lambda)$, and resolvent $R_V(\lambda + i0)$ are related by the well-known formula

$$\langle E_{ac}(d\lambda)f, g \rangle = \left\langle \frac{1}{2\pi i} [R_V(\lambda + i0) - R_V(\lambda - i0)] f, g \right\rangle d\lambda.$$

Since $\chi_L(H)E(d\lambda) = \chi_L(H)E_{ac}(d\lambda)$ one concludes that

$$|\langle e^{itH} \chi_L(H)f, g \rangle| = \left| (2\pi i)^{-1} \sum_{k \geq 0} \int_{-\infty}^{+\infty} e^{it\lambda^2} \chi(\lambda^2) \lambda \langle R_0(\lambda^2 + i0)(VR_0(\lambda^2 + i0))^k f, g \rangle d\lambda \right|$$

where we have first changed variables $\lambda \rightarrow \lambda^2$.

The kernel of $R_0(\lambda^2 + i0)(VR_0(\lambda^2 + i0))^k$ is given by the formula

$$\mathcal{K}(x, y, \lambda^2 + i0) = \int_{\mathcal{R}^k} \left(\prod_{j=1}^k V(x_j) \right) K(x, x_1, \lambda^2 + i0) K(x_k, y, \lambda^2 + i0) K(x_1, x_2, \lambda^2 + i0) \dots K(x_{k-2}, x_{k-1}, \lambda^2 + i0) dx_1 \dots dx_k.$$

This allows to apply Fubini's theorem and obtain

$$|\langle e^{itH} \chi_L(H)f, g \rangle| \leq \sum_{k \geq 0} \left(\frac{N-1}{N\sqrt{\lambda_0}} \right)^k \sup_{a \in \mathbb{R}} \left| \int_{-\infty}^{+\infty} e^{i(t\lambda^2 + a\lambda)} \chi_L(\lambda^2) \lambda^{-k} \lambda_0^{k/2} d\lambda \right| \|V\|_1^k \|f\|_1 \|g\|_1.$$

Since the quantity inside the absolute values is the solution of the free Schrödinger operator on \mathbb{R} at time t and position a , the dispersive bound for this operator yields (or simply applying Parseval identity, see [10, p. 161] for more details)

$$|\langle e^{itH} \chi_L(H)f, g \rangle| \leq C(V) |t|^{-1/2} \|f\|_1 \|g\|_1,$$

where $C(V) = 2 \sup_{k \geq 0} \sup_{L \geq 1} \left\| \mathcal{F}^{-1} \left[\chi_L(\lambda^2) \lambda^{-k} \lambda_0^{k/2} \right] \right\|_1$ that is finite due to the arguments from [10, p. 161]. \square

5.2 Low energy estimate

For any smooth and compactly supported cut-off function χ in \mathbb{R} , by Theorem 4.15 we have for any $x, x' \in R$

$$2i\pi \int_0^{+\infty} e^{it\lambda} \chi(\lambda) E_{ac}(d\lambda)(x, x') = -2i\pi \int_0^{+\infty} e^{it\lambda} \chi(\lambda) \Im K(x, x', \lambda) d\lambda,$$

and by the change of variables $\lambda = \mu^2$, we get

$$2i\pi \int_0^{+\infty} e^{it\lambda} \chi(\lambda) E_{ac}(d\lambda)(x, x') = -4i\pi \int_0^{+\infty} e^{it\mu^2} \chi(\mu^2) \Im K(x, x', \mu^2) \mu d\mu.$$

Now recalling the definition of K , we again distinguish between the following three cases:

1. If $x, x' \in R_j$ with $x' > x$, then

$$K(x, x', \mu^2) = \frac{1}{W_{j,-}(\mu)} f_{j,-}(\mu, x) f_{j,+}(\mu, x') + \frac{c_{j,-,2}(\mu)}{i\mu f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{j,+}(\mu, x').$$

As $\overline{f_{j,\pm}(\mu, x)} = f_{j,\pm}(-\mu, x)$, we deduce

$$\begin{aligned} 2i\pi \int_0^{+\infty} e^{it\lambda} \chi(\lambda) E_{ac}(d\lambda)(x, x') &= -2i\pi \int_{-\infty}^{+\infty} e^{it\mu^2} \chi(\mu^2) \mu \frac{1}{W_{j,-}(\mu)} f_{j,-}(\mu, x) f_{j,+}(\mu, x') d\mu \\ &\quad - 2i\pi \int_{-\infty}^{+\infty} e^{it\mu^2} \chi(\mu^2) \frac{c_{j,-,2}(\mu)}{f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{j,+}(\mu, x') d\mu. \end{aligned}$$

The first term of this right hand side was estimated in Lemma 4 of [10], hence it remains to estimate the second term. For that purpose, we set

$$\begin{aligned} T_2(t, x, x') &:= \int_{-\infty}^{+\infty} e^{it\mu^2} \chi(\mu^2) \frac{c_{j,-,2}(\mu)}{f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{j,+}(\mu, x') d\mu \\ &= \int_{-\infty}^{+\infty} e^{it\mu^2} \chi(\mu^2) e^{i\mu(x+x')} \frac{c_{j,-,2}(\mu)}{f_{j+1,+}(\mu, 0) s(\mu)} m_{j,+}(\mu, x) m_{j,+}(\mu, x') d\mu. \end{aligned}$$

Hence denoting by

$$p(\mu) := \frac{c_{j,-,2}(\mu)}{f_{j+1,+}(\mu, 0) s(\mu)},$$

we have shown in Corollary 4.10 that this function belongs to $H^1(-R, R)$, for all $R > 0$. Since the mapping

$$q : \mu \rightarrow \chi(\mu^2) m_{j,+}(\mu, x) m_{j,+}(\mu, x'),$$

has compact support and is in $C^1(\mathbb{R})$ with the property

$$|q(\mu)| + |\dot{q}(\mu)| \leq C,$$

for some $C > 0$ independent of x and x' due to (4.26) and (4.42), we deduce that the product pq belongs to $H^1(\mathbb{R})$. By Plancherel theorem (see for instance [16, p. 60]), we deduce that

$$T_2(t, x, x') = t^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \mathcal{F}(pq)(\xi + x + x') e^{-\frac{i\xi^2}{t}} d\xi.$$

and consequently

$$\begin{aligned} |T_2(t, x, x')| &\leq |t|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} |\mathcal{F}(pq)(\xi + x + x')| d\xi \\ &\leq |t|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} |\mathcal{F}(pq)(\xi)| d\xi \\ &\leq C |t|^{-\frac{1}{2}} \|pq\|_{H^1(\mathbb{R})}. \end{aligned}$$

for some $C > 0$.

2. If $x, x' \in R_j$ with $x' < x$, then

$$K(x, x', \mu^2) = \frac{1}{W_{j,-}(\mu)} f_{j,-}(\mu, x') f_{j,+}(\mu, x) + \frac{c_{j,-,2}(\mu)}{i\mu f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{j,+}(\mu, x').$$

In that case the first term was treated in Lemma 4 of [10], while the second term is the same as before.

2. If $x \in R_j$ and $x' \in R_k$ with $k \neq j$, then

$$K(x, x', \mu^2) = \frac{1}{i\mu f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{k,+}(\mu, x').$$

Therefore in that case we have

$$2i\pi \int_0^{+\infty} e^{it\lambda} \chi(\lambda) E_{ac}(d\lambda)(x, x') = 2\pi \int_{-\infty}^{+\infty} e^{it\mu^2} \chi(\mu^2) \frac{1}{f_{j+1,+}(\mu, 0) s(\mu)} f_{j,+}(\mu, x) f_{j,+}(\mu, x') d\mu.$$

Since $\frac{1}{f_{j+1,+}(\mu, 0)}$ is in $C^1(\mathbb{R})$, by Corollary 4.9, the function

$$\mu \rightarrow \frac{1}{f_{j+1,+}(\mu, 0) s(\mu)}$$

belongs to $H^1(-R, R)$, for all $R > 0$ and we conclude as for $T_2(t, x, x')$.

□

References

- [1] F. Ali Mehmeti, Spectral Theory and L^∞ -time Decay Estimates for Klein-Gordon Equations on Two Half Axes with Transmission: the Tunnel Effect. *Math. Methods Appl. Sci.* **17** (1994), 697–752.
- [2] F. Ali Mehmeti, Transient Waves in Semi-Infinite Structures: the Tunnel Effect and Sommerfeld Problem. Mathematical Research, vol. 91, Akademie Verlag, Berlin, 1996.
- [3] F. Ali Mehmeti, R. Haller-Dintelmann and V. Régner, Expansions in generalized eigenfunctions of the weighted laplacian on star-shaped networks, *Functional analysis and evolution equations*, 1-16, Birkhäuser, Basel, 2008.
- [4] F. Ali Mehmeti, R. Haller-Dintelmann and V. Régner, Multiple tunnel effect for dispersive waves on a star-shaped network: an explicit formula for the spectral representation, arXiv:1012.3068v1 [math.AP], Preprint 2010, *submitted*.
- [5] F. Ali Mehmeti, R. Haller-Dintelmann, V. Régner, The influence of the tunnel effect on L^∞ -time decay, *Operator Theory: Advances and Applications*, **221** (2012), 11-24.
- [6] F. A. Berzin and M. A. Shubin, The Schrödinger equation, Kluwer Academic Publishers, 1991.
- [7] A. V. Borovskikh, Method of Propagating waves for a one-dimensional inhomogeneous medium, *Journal of Mathematical Sciences*, **127**,5 (2005), 2135-2158.
- [8] T. Cazenave, Semilinear Schrödinger equations, Courant Lect. Notes Math. 10, American society, Providence, RI, Courant Institute of Mathematical Sciences, New york, 2003.
- [9] P. Deift and E. Trubowitz, Inverse scattering on the line, *Comm. Pure Appl. Math.*, **XXXII** (1979), 121-251.
- [10] M. Goldberg and W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, *Commun. Math. Phys.*, **251** (2004), 157-178.
- [11] L. Ignat, Strichartz estimates for the Schrödinger equation on a tree and applications, *SIAM. J. Math. Anal.*, **42** (2010), 2041-2057.
- [12] T. Kato, *Perturbation theory for linear operators*. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
- [13] M. Kell and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.*, **120** (1998), 955-980.

- [14] S. Nicaise, Diffusion sur les espaces ramifiés, Thesis, Université de l'Etat à Mons, 1986.
- [15] R. Racke, Lectures on nonlinear evolution equations. Initial value problems, Aspects of Mathematics E19, Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1992.
- [16] M. Reed and B. Simon, Methods of modern mathematical physics II: Self-adjointness, Academic Press, 1975.
- [17] M. Reed and B. Simon, Methods of modern mathematical physics III: Scattering theory, Academic Press, 1979.
- [18] R. Weder, $L^p - L^{p'}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, *Journal of Functional Analysis*, **170** (2000), 37-68.